Notes of 2.4 (error analysis for iterative methods)

Monday, February 8, 2021 5:32 PM

Def

p_n converges to p of order $\alpha > 0$ with asymptotic error constant $\lambda > 0$ if:

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

(need $p_n \neq p$ for all n)

Def

if $\alpha = 1$ and $\lambda < 1$, then this is **linearly convergent**
if $\alpha = 2$, this is **quadratically convergent**.

Theorem 2.8 (linear convergence of fixed point method)

If g is continuous on $[a, b]$, g' continuous on (a, b), $|g'(x)| < 1$ on (a, b), $g'(p) \neq 0$, then $p_n = g(p_{n-1})$ converges linearly to the unique fixed point in $[a, b]$
(unless I am mistaken, really we need to possibly shrink our interval to insure mapping properties).

proof

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n\to\infty} \frac{g'(\xi_n)(p_n - p)}{p_n - p} = \lim_{n\to\infty} g'(\xi_n) = g'(p)$$

take absolute values of everything, get $\alpha = 1, \lambda = g'(p)$

Theorem 2.9 (quadratic convergence of fixed point method)

If $g(p) = p, g'(p) = 0, |g''(x)| < M$ around p, then if p_0 is near p, then we get $|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$

Proof.

Taylor expand around p: $g(x) = p + \frac{(x-p)^2}{2} g''(\xi)$
therefore $p_{n+1} = g(p_n) = p_n + \frac{g''(\xi)(p-p_n)^2}{2}$
therefore $p_{n+1} - p_n = \frac{g''(\xi)(x-p_n)^2}{2}$
therefore $\frac{|p_{n+1}-p_n|}{|p-p_n|^2} = \frac{|g''(\xi)|}{2} \to \frac{|g''(p)|}{2} \leq \frac{2}{M}$

Conclusion

Fixed points converge fast if $g'(p) = 0$

To find good fixed point

If we want to solve $f(x) = 0$
we could let $g(x) = x - f(x)$, but a better one would be $g(x) = x - \phi(x)f(x)$ where $\phi(x)$ is some function such that $g'(p) = 0$.

after easy math, turns out $\phi(x) = 1/f'(x)$ which is Newton's method.

Therefore:
Theorem. (Convergence of Newton's method)
Newton's method converges quadratically

Def (multiplicity of a root)
A root of f at p has multiplicity m if $f(x) = (x-p)^m q(x)$ with $\lim_{x \to p} q(x) \neq 0$ (basically you can factor $(x-p)^m$ out)

Criterion for multiplicity of a root.
if $f \in C^k([a,b])$ then $f^{(i)}(p) = 0$ for $i = 0, \dots, k-1$, $f^{(k)}(p) \neq 0$ if and only if f has a zero of multiplicity k